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Zeros on the temperature axis of spin correlations in a random bond Ising chain

M Nifle and H J Hilhorst

Laboratoire de Physique Théorique et Hautes Energies†, Bâtiment 211, Université de Paris-Sud, 91405 Orsay, France

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Abstract. We study the correlation between two spins diametrically opposite in a random Ising loop of $2r$ spins with random nearest-neighbour couplings. As a function of the inverse temperature β, this correlation undergoes random sign changes with a density $\rho_r(\beta)$. We determine this density for an arbitrary distribution $P(K)$ of the coupling constants for (i) $r\to\infty$ and (ii) the scaling limit $\beta\to\infty$, $r\to\infty$ at r/β fixed. For $r\to\infty$, the total number of zeros on the β-axis grows asymptotically as

$$\frac{1}{2}\left[\frac{1}{21\zeta(3)}-\frac{1}{4\pi^2}\right]^{1/2}\log r.$$

As an application, the density of zeros is calculated for the spin–spin correlation in a double infinite Ising chain with random bonds, having, with a probability p, an infinite transverse coupling between each pair of corresponding sites. A connection is made with predictions from spin glass theory.

1. Introduction

We consider the most elementary of all frustrated spin systems, namely a loop of $2r$ Ising spins $s_1, s_2, \ldots, s_{2r}$, having random nearest-neighbour couplings $K_1, K_2, \ldots, K_{2r}$ distributed symmetrically around zero. Hence, the system is frustrated with probability $\frac{1}{2}$, and our interest is in these frustrated loops. We shall consider a zero field spin–spin correlation function in this loop, and ask how, with a fixed distance between the spins, this quantity changes with temperature.

In spite of the simplicity of the system under consideration, this question is non-trivial. Its motivation comes from higher-dimensional systems, and we shall briefly describe it first. In the low-temperature phase of a spin glass, the value of the spin–spin correlation function, at fixed distance r, is hypersensitive to external parameters, so that one may say that it varies randomly with them. This has been shown in mean-field theory by Parisi [1], in a Gaussian approximation around mean-field theory by Kondor [2], and on the basis of a droplet model approximation in finite dimensions by Fisher and Huse [3, 4] and Bray and Moore [5]. The random temperature dependence of the equilibrium state is of interest for its own sake, and also because it has profound consequences for spin glass dynamics. As shown by Koper and Hilhorst [6, 7] and by Fisher and Huse [8], it is in particular essential for the explanation of aging phenomena [9–11] in small magnetic fields.

† Laboratoire associé au Centre National de la Recherche Scientifique.

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Lack of knowledge of the precise random properties of the equilibrium correlation functions has recently led Koper and Hilhorst [12, 13] and the present authors [14] to postulate such properties and to proceed from there to study the consequences for the dynamics. However, it is desirable to have some exact results on the temperature dependence of the equilibrium correlation functions in frustrated systems. Phenomena that appear in the spin glass phase of a true spin glass are also expected to occur at low temperatures in a frustrated one-dimensional system, on a length scale less than the correlation length, in spite of the absence of a phase transition.

One fundamental quantity that characterizes the random temperature dependence of a correlation function is its density of zeros on the temperature axis, obtained after averaging over all random couplings. In this work we calculate this density, called $\rho_r(\beta)$, for the correlation $\langle s_0 s_r\rangle_\beta$, at inverse temperature β, between two spins diametrically opposite on the loop. The distribution law $P(K)$ of the couplings is symmetric in K, sufficiently smooth with $P(0)\neq 0$, but otherwise arbitrary.

In section 2 we formulate the problem mathematically. In section 3 we first consider the large-r limit and find a simple and general expression for $\rho_\infty(\beta)$ in terms of averages with respect to P. This expression is analysed for its large-β and small-β behaviour, and the special case of a double-peaked Gaussian distribution is worked out.

In sections 4 and 5 we consider $\rho_r(\beta)$ in the scaling limit $r, \beta\to\infty$ with r/β constant. In section 4, we obtain a scaling function of the variable $\alpha\equiv rP(0)/\beta$, for $\alpha>\frac{1}{2}$. In section 5, we study the expansion of the scaling function when $\alpha\ll 1$, and show that in this limit

$$\rho_r(\beta)\sim\frac{1}{\beta}\left(\frac{rP(0)}{\beta}\right)^2.$$

A physical interpretation of this limit is given.

In section 6 we calculate the total number of zeros of $\langle s_0 s_r\rangle_\beta$ on the β axis, and show that as $r\to\infty$ it increases as

$$\frac{1}{2}\left[\frac{1}{21\zeta(3)}-\frac{1}{4\pi^2}\right]^{1/2}\cdot\log r.$$

In section 7 we discuss, as an application, the density of zeros of the correlation function in a double Ising chain with random intrachain couplings, and with interchain couplings that randomly take the values $\pm\infty$ (each with probability $\frac{1}{2}p$) or 0.

In section 8 we indicate possible extensions of our results. We also indicate the relation to a quantity often considered in other works, namely the overlap length between two thermodynamic equilibrium states at different temperatures.

2. A frustrated loop of Ising spins

2.1. The correlation function

We consider a loop of $2r$ Ising spins described by the Hamiltonian

$$\mathscr{H}=\sum_{i=1}^{2r}K_i s_{i-1}s_i \qquad s_0\equiv s_{2r}. \tag{2.1}$$

The s_i may take the values ± 1, and the $2r$ coupling constants K_i are identically distributed independent quenched random variables, with a probability law $P(K_i)$

symmetric about zero. Therefore, the loop is frustrated with probability $\frac{1}{2}$, and we are interested in this subclass of frustrated loops.

We shall consider the correlation function $\langle s_0 s_r \rangle_\beta$ between two spins opposite to each other on the loop. Here $\langle \ldots \rangle_\beta$ is the thermal average at inverse temperature β, for a fixed set $\{K_i\}$. The simplicity of this system allows one to write the explicit result

$$\langle s_0 s_r \rangle_\beta = \frac{T_1(\beta) + T_2(\beta)}{1 + T_1(\beta) T_2(\beta)} \tag{2.2}$$

where

$$T_1(\beta) = \prod_{i=1}^{r} \tanh \beta K_i \qquad T_2(\beta) = \prod_{i=r+1}^{2r} \tanh \beta K_i. \tag{2.3}$$

We wish to consider $\langle s_0 s_r \rangle_\beta$ *as a function of temperature.*

2.2. *The density of zeros of* $\langle s_0 s_r \rangle_\beta$

It is easy to see that when the loop is not frustrated, then $T_1(\beta)$ and $T_2(\beta)$, and hence $\langle s_0 s_r \rangle_\beta$, will be of a single sign for all β. For a frustrated chain, however, temperatures β_ν may occur at which the correlation $\langle s_0 s_r \rangle_\beta$ vanishes.

These β_ν are the solutions of

$$f_r(\beta) \equiv T_1(\beta) + T_2(\beta) = 0. \tag{2.4}$$

Their number and their values depend on the disorder variables $\{K_i\}$. We define $\rho_r(\beta)\,\mathrm{d}\beta$ as the number of zeros in an interval $\mathrm{d}\beta$ after averaging over all $\{K_i\}$. In formula,

$$\rho_r(\beta) = \overline{\sum_\nu \delta(\beta - \beta_\nu)} = \overline{\left| \frac{\mathrm{d}f_r(\beta)}{\mathrm{d}\beta} \right| \delta(f_r(\beta))} \tag{2.5}$$

where the overbar indicates the disorder average. From (2.3), (2.4) and (2.5), we have

$$\rho_r(\beta) = \overline{|T_1(\beta) G_1(\beta) + T_2(\beta) G_2(\beta)|\, \delta(T_1(\beta) + T_2(\beta))} \tag{2.6}$$

in which

$$G_1(\beta) = \sum_{i=1}^{r} g(\beta K_i) \qquad G_2(\beta) = \sum_{i=r+1}^{2r} g(\beta K_i) \tag{2.7}$$

$$g(x) = \frac{2x}{\sinh 2x}. \tag{2.8}$$

It is convenient to introduce

$$\begin{aligned} P_+(K_i) &= 2P(K_i) \qquad && \text{if } K_i \geq 0 \\ &= 0 && \text{if } K_i < 0. \end{aligned} \tag{2.9}$$

Taking into account the symmetries of the problem we can then write $\rho_r(\beta)$ as

$$\rho_r(\beta) = \frac{1}{2\beta} \overline{|G_1(\beta) - G_2(\beta)| \delta(\log T_1(\beta) - \log T_2(\beta))} \tag{2.10}$$

where, here and henceforth, the overbar indicates the average over all K_i with respect to the distribution $P_+(K_i)$, and where the extra factor $\frac{1}{2}$ is the probability for the loop to be frustrated.

This formula will be the starting point for the analysis of section 5. Upon introducing another delta function it becomes

$$\rho_r(\beta)=\frac{1}{2\beta}\int_{-\infty}^{+\infty}\mathrm{d}x\,|x|\,\overline{\delta(x-G_1(\beta)+G_2(\beta))\delta(\log T_1(\beta)-\log T_2(\beta))}. \tag{2.11}$$

Finally, we make use of integral representations of the delta functions in (2.11). The $2r$ integrations on the K_i then decouple and give $2r$ identical factors, and the result is

$$\rho_r(\beta)=\frac{1}{2\beta}\int_{-\infty}^{+\infty}\mathrm{d}x\,|x|\int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu}{2\pi}\,\mathrm{e}^{\mathrm{i}\lambda x}\,|F(\lambda,\mu)|^{2r} \tag{2.12}$$

where

$$F(\lambda,\mu)=\overline{\exp[\mathrm{i}\lambda g(\beta K)+\mathrm{i}\mu h(\beta K)]} \tag{2.13}$$

and

$$h(x)=-\log\tanh x. \tag{2.14}$$

Equation (2.12) is an alternative expression for the density of zeros of the correlation function, which we shall use in sections 3 and 4.

3. The limit of a large loop ($r\gg 1$ with β constant)

3.1. The result for general β

It is first of interest to consider equation (2.12) in the limit $r\gg 1$, i.e. to calculate $\rho_\infty(\beta)$. We expand (2.13) in powers of λ and μ, which gives

$$F(\lambda,\mu)=1+\sum_{m=1}^{\infty}\frac{1}{m!}\overline{(\mathrm{i}\lambda g(\beta K)+\mathrm{i}\mu h(\beta K))^m}. \tag{3.1}$$

Then we re-exponentiate and find

$$F(\lambda,\mu)=\exp[\mathrm{i}\lambda\bar{g}+\mathrm{i}\mu\bar{h}-\tfrac{1}{2}(\lambda^2\overline{\Delta g^2}+2\lambda\mu\overline{\Delta g\Delta h}+\mu^2\overline{\Delta h^2}+\ldots] \tag{3.2}$$

where

$$\Delta f(\beta K)\equiv f(\beta K)-\overline{f(\beta K)}\qquad (f=g,h). \tag{3.3}$$

We now use (3.2) in (2.12). The quantity $|F(\lambda,\mu)|^{2r}$ is an exponential that no longer contains linear terms in λ and μ. After introducing the scaled integration variables $\tilde{\lambda}=\lambda\sqrt{r}$, $\tilde{\mu}=\mu\sqrt{r}$, and $\tilde{x}=x/\sqrt{r}$, one can straightforwardly expand the integrand in powers of r^{-1}. The leading term is a constant independent of r. Carrying out the integrals, which are all Gaussian, we obtain from (2.12)

$$\rho_\infty(\beta)=\frac{1}{2\pi\beta}\left(\frac{\overline{\Delta g^2}\,\overline{\Delta h^2}-\overline{\Delta g\Delta h}^2}{\overline{\Delta h^2}^2}\right)^{1/2} \tag{3.4}$$

with g and h given by (2.8) and (2.14), respectively. The disorder averages in (3.4) still depend on β. We shall consider successively the limits $\beta\gg 1$ and $\beta\ll 1$ in the next subsection.

The result (3.4) is the first term in the large-r expansion of $\rho_r(\beta)$, which can be written as

$$\rho_r(\beta)=\rho_\infty(\beta)+\mathrm{O}(r^{-1}). \tag{3.5}$$

3.2. Large-β and small-β limits of $\rho_\infty(\beta)$

(i) The limit $\beta \gg 1$. We perform a large-β expansion of the averages in (3.4). For an arbitrary function f of the variable βK, one has

$$\bar{f} = \int_0^{+\infty} \mathrm{d}K \, P_+(K) f(\beta K). \tag{3.6}$$

After taking βK as the new variable of integration, we expand P_+ around the origin and find

$$\bar{f} = \frac{1}{\beta} \int_0^{+\infty} \mathrm{d}x \, f(x) \left[P_+(0) + \mathrm{O}\left(\frac{1}{\beta}\right) \right] \tag{3.7}$$

provided that $f(x)$ is integrable. We obtain with the aid of (3.3) and (3.7)

$$\overline{\Delta h^2} = \frac{P_+(0)}{\beta} \int_0^{+\infty} \mathrm{d}x \, h^2(x) + \mathrm{O}\left(\frac{1}{\beta^2}\right) \tag{3.8a}$$

$$\overline{\Delta g^2} = \frac{P_+(0)}{\beta} \int_0^{+\infty} \mathrm{d}x \, g^2(x) + \mathrm{O}\left(\frac{1}{\beta^2}\right) \tag{3.8b}$$

$$\overline{\Delta g \Delta h} = \frac{P_+(0)}{\beta} \int_0^{+\infty} \mathrm{d}x \, g(x) h(x) + \mathrm{O}\left(\frac{1}{\beta^2}\right). \tag{3.8c}$$

We substitute (3.8) in (3.4). To leading order, the term in brackets in (3.4) is a numerical constant, denoted c^2 henceforth, and we have

$$\rho_\infty(\beta) = \frac{c}{2\pi\beta} + \mathrm{O}\left(\frac{1}{\beta^2}\right). \tag{3.9a}$$

In appendix A we have evaluated c with the result

$$c^2 = \pi^2/(21\zeta(3)) - \tfrac{1}{4} = 0.14098\ldots \tag{3.9b}$$

(ii) The limit $\beta \ll 1$. We make the small-β expansion of the functions (2.8) and (2.14)

$$g(\beta K) \simeq 1 - \tfrac{2}{3}(\beta K)^2 \tag{3.10a}$$

$$h(\beta K) \simeq -\log \beta K. \tag{3.10b}$$

With the aid of these one finds that, to leading order,

$$\begin{aligned} \overline{\Delta h^2} &\simeq \overline{\log K^2} - \overline{\log K}^2 \\ \overline{\Delta g^2} &\simeq \tfrac{4}{9}\beta^4(\overline{K^4} - \overline{K^2}^2) \\ \overline{\Delta h \Delta g} &\simeq \tfrac{2}{3}\beta^2(\overline{K^2 \log K} - \overline{K^2}\,\overline{\log K}). \end{aligned} \tag{3.11}$$

Upon substituting (3.11) in (3.4), one sees that

$$\rho_\infty(\beta) \simeq C\beta \qquad \text{for } \beta \ll 1 \tag{3.12}$$

where C is a constant composed of averages of powers and logarithms of K.

The result (3.4) for $\rho_\infty(\beta)$, as well as the results (3.9) and (3.12) for the two limits $\beta \ll 1$ and $\beta \gg 1$, hold for an arbitrary probability law $P_+(K)$ for the coupling constants. In the next subsection, we shall consider a specific choice for $P_+(K)$.

3.3. A double Gaussian distribution

If for $P(K)$ we take the sum of two Gaussians centred around $\pm K_0$, we get for $P_+(K)$

$$P_+(K)=\frac{1}{\sqrt{2\pi\sigma^2}}\left(\exp\left(-\frac{(K+K_0)^2}{2\sigma^2}\right)+\exp\left(-\frac{(K-K_0)^2}{2\sigma^2}\right)\right) \qquad K\geqslant 0. \tag{3.13}$$

In the limit of a narrow width,

$$\sigma \ll 1 \tag{3.14}$$

the function $\rho_\infty(\beta)$ can be calculated explicitly, as we shall now show.

In this limit we may neglect the contribution to $P_+(K)$ of the peak at $K=-K_0$. Then we make use of the saddle-point method to write the average of an arbitrary function $G(K)$ as

$$\bar{G}=G_0+\frac{\sigma^2}{2}G_0''+\frac{\sigma^4}{8}G_0^{IV}+O(\sigma^6) \tag{3.15}$$

where the subscript 0 indicates evaluation for $K=K_0$. If $H(K)$ is another arbitrary function, and if $\Delta G(K)\equiv G(K)-\bar{G}$ and $\Delta H(K)\equiv H(K)-\bar{H}$, then one derives from (3.15) the relation

$$\begin{aligned}\overline{\Delta G\Delta H}&=\overline{GH}-\bar{G}\bar{H}\\ &=\sigma^2G_0'H_0'+\tfrac{1}{2}\sigma^4(G_0'''H_0'+G_0''H_0''+G_0'H_0''')+O(\sigma^6).\end{aligned} \tag{3.16}$$

By substituting the functions $g(\beta K)$ and $h(\beta K)$ for $G(K)$ and $H(K)$, it is possible to find the saddle-point expansions of all averages required in (3.4). After inserting these in (3.4) one finds

$$\rho_\infty(\beta)=\frac{\sigma}{\pi\sqrt{2}}\left(\frac{g'}{h'}\right)'_{K=K_0}+O(\sigma^3). \tag{3.17}$$

Using the explicit expressions (2.8) and (2.14) for g and h, respectively, one arrives at

$$\rho_\infty(\beta)\simeq\frac{1}{\sqrt{2}\,\pi}\,\frac{\sinh 4\beta K_0-4\beta K_0}{\cosh 4\beta K_0-1}\,\sigma \qquad \text{for } \sigma\ll 1,\ \beta \text{ fixed}. \tag{3.18}$$

For $\beta\gg 1$, the coefficient of σ in (3.18) tends towards a constant, and hence (3.18) is *not integrable. The reason is that the limits* $\sigma\ll 1$ *and* $\beta\gg 1$ *do not commute.*

4. The scaling limit r, $\beta\gg 1$ with r/β constant. I

In the sector of the (r,β)-plane where r, $\beta\gg 1$ with r/β fixed, $\rho_r(\beta)$ can be written with the help of a scaling function of the variable $\alpha\equiv rP(0)/\beta$. The starting point that yields this result, at least for $\alpha>\frac{1}{2}$, is provided by equation (2.12) with equation (2.13) rewritten as

$$F(\lambda,\mu)=1+\overline{(\exp[\mathrm{i}\lambda g(\beta K)+\mathrm{i}\mu h(\beta K)]-1)}. \tag{4.1}$$

We expand the argument of the average in powers of $1/\beta$, using (3.8), and get after re-exponentiating

$$F(\lambda,\mu)=\exp\left(\frac{2P(0)}{\beta}\int_0^{+\infty}\mathrm{d}y(\exp[\mathrm{i}\lambda g(y)+\mathrm{i}\mu h(y)]-1)+O\left(\frac{1}{\beta^2}\right)\right). \tag{4.2}$$

Then the integrand of (2.12) becomes

$$|F(\lambda,\mu)|^{2r}=\exp(-8(rP(0)/\beta)c(\lambda,\mu)) \tag{4.3}$$

where

$$c(\lambda,\mu)=\int_0^{+\infty}\mathrm{d}y\,\sin^2(\tfrac{1}{2}\lambda g(y)+\tfrac{1}{2}\mu h(y)). \tag{4.4}$$

Upon substituting this in (2.12) we find

$$\rho_r(\beta)=\frac{1}{\beta}\mathscr{F}_+\left(\frac{rP(0)}{\beta}\right) \tag{4.5}$$

with the scaling function $\mathscr{F}_+$ given by

$$\mathscr{F}_+(\alpha)=\tfrac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}x\,|x|\int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{2\pi}\exp(\mathrm{i}\lambda x)\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu}{2\pi}\exp(-8\alpha c(\lambda,\mu)). \tag{4.6}$$

Two comments should be made regarding the result (4.6).

First, the requirement that these integrals converge imposes the condition

$$\alpha>\tfrac{1}{2} \tag{4.7}$$

as we shall show now by studying the behaviour of $c(\lambda,\mu)$. For large λ and large μ, the integrand in (4.4) oscillates rapidly in y around an average of $\frac{1}{2}$, until, for $y\gtrsim y_0(\lambda,\mu)$, it decreases exponentially to zero. The point y_0 is determined by

$$|\lambda g(y)+\mu h(y)|\lesssim 1. \tag{4.8}$$

Upon using the large y expansion of g and h, the condition (4.8) yields, to leading order as $|\lambda|$ or $|\mu|$ gets large, that y_0 grows as

$$y_0(\lambda,\mu)\simeq\tfrac{1}{2}\max(\tfrac{1}{2}\log|\lambda|,\tfrac{1}{2}\log|\mu|). \tag{4.9}$$

Therefore

$$c(\lambda,\mu)\simeq\tfrac{1}{2}y_0(\lambda,\mu) \tag{4.10}$$

and the integrand in (4.6) behaves as

$$\exp(-8\alpha c(\lambda,\mu))\sim\min\left(\frac{1}{|\lambda|^{2\alpha}},\frac{1}{|\mu|^{2\alpha}}\right) \tag{4.11}$$

in the large-λ and large-μ limit. This gives condition (4.7).

Second, one can study the scaling function (4.6) in the large-α limit. To this end, we perform a small-λ and small-μ expansion of (4.4), which gives

$$c(\lambda,\mu)=\tfrac{1}{4}(c_0\lambda^2+2c_1\lambda\mu+c_2\mu^2)+\ldots \tag{4.12}$$

where

$$c_n=\int_0^{+\infty}\mathrm{d}y\,g^n(y)h^{2-n}(y)\qquad n=0,1,2. \tag{4.13}$$

After substitution of this expression in (4.6) and introduction of the scaled variables of integration $\tilde{\lambda}=\lambda\sqrt{\alpha}$, $\tilde{\mu}=\mu\sqrt{\alpha}$ and $\tilde{x}=x/\sqrt{\alpha}$, the integrand can be expanded in inverse powers of α and integrated term by term. The integrals are all Gaussian and we find

$$\mathscr{F}_+(\alpha)=\frac{1}{2\pi}\left(\frac{c_2c_0-c_1^2}{c_0^2}\right)^{1/2}+O\left(\frac{1}{\alpha}\right)\qquad\alpha\gg1. \tag{4.14}$$

This result is identical to that obtained in section 3.2, where we have first taken the large-r limit and then the large-β limit.

5. The scaling limit r, $\beta \gg 1$ with r/β constant. II

In the sector of the (r, β)-plane not covered by the preceding section, i.e. where $r \gg 1$, $\beta \gg 1$ and $\alpha \equiv rP(0)/\beta < \frac{1}{2}$, the function $\rho_r(\beta)$ can be cast in a similar scaling form, namely

$$\rho_r(\beta) = \frac{1}{\beta} \mathscr{F}_-\left(\frac{rP(0)}{\beta}\right) \tag{5.1}$$

where $\mathscr{F}_-(\alpha)$ can be found as a power series in $\alpha \equiv rP(0)/\beta$. We shall show that only three coupling constants constants contribute to the leading order result.

Upon using (2.3) and (2.7), the equation (2.10) takes the form

$$\rho_r(\beta) = \frac{1}{2\beta} \overline{\left| \sum_{i=1}^{r} g(\beta K_i) - \sum_{i=r+1}^{2r} g(\beta K_i) \right| \delta\left(\sum_{i=1}^{r} h(\beta K_i) - \sum_{i=r+1}^{2r} h(\beta K_i) \right)}. \tag{5.2}$$

In order to perform the K_i-integrals implicit in (5.2), we shall use as new variables of integration the lengths of the intervals into which the K_i divide the positive K-axis. Let these interval lenths, from left to right, be denoted $\Delta_1, \Delta_2, \ldots, \Delta_{2r}$. Their joint probability density, to be called $\mathscr{P}$, can be derived from the $P_+(K_i)$. Equation (5.2) then becomes

$$\begin{aligned}\rho_r(\beta) = \frac{1}{2\beta} \int_0^{+\infty} \mathrm{d}\Delta_1 \ldots \int_0^{+\infty} \mathrm{d}\Delta_{2r}\, \mathscr{P}(\Delta_1, \ldots, \Delta_{2r}) \\ \times [|\varepsilon_1 g(\beta\Delta_1) + \varepsilon_2 g(\beta\Delta_1 + \beta\Delta_2) + \ldots| \\ \times \delta[\varepsilon_1 h(\beta\Delta_1) + \varepsilon_2 h(\beta\Delta_1 + \beta\Delta_2) + \ldots]_{\mathrm{av}}\end{aligned} \tag{5.3}$$

Here the signs ε_i should be taken from a set of r plus signs and r minus signs, and in the end one has to average over all ways of doing this, as indicated by $[\ldots]_{\mathrm{av}}$. The dots that occur twice inside these brackets in (5.3), indicate the $2r-2$ remaining terms.

We turn now to the limit $r \gg 1$ and $\beta \gg 1$. If Δ_1 remains of order β^0 when $\beta \gg 1$, then one may use the large-β expansion for all gs and hs in (5.3). The delta function constraint in (5.3) then takes the form

$$\varepsilon_1 + \varepsilon_2\, \mathrm{e}^{-2\beta\Delta_2} + \varepsilon_3\, \mathrm{e}^{-2\beta(\Delta_2+\Delta_3)} + \varepsilon_4\, \mathrm{e}^{-2\beta(\Delta_2+\Delta_3+\Delta_4)} + \ldots = 0. \tag{5.4}$$

Since each next term on the left-hand side of (5.4) is smaller in absolute value than its predecessor, the constraint can be satisfied, in the large-β limit, only if at least Δ_2 and Δ_3 are of the order $1/\beta$. For each interval Δ_i which is of order $1/\beta$, the phase-space volume in Δ-space will be reduced by a factor $1/\beta$. We therefore expect the leading contribution to (5.3) to come from

$$\Delta_2, \Delta_3 \text{ of order } 1/\beta \qquad \Delta_1 \text{ and } \Delta_4, \ldots, \Delta_{2r} \text{ of order } 1. \tag{5.5}$$

If we take the limit $\beta \gg 1$ subject to (5.5), the integrand in (5.3) no longer depends on Δ_1 and $\Delta_4, \ldots, \Delta_{2r}$ so that we only need the marginal law for Δ_2 and Δ_3 that follows from $\mathscr{P}$. In the large-r limit, these are identically distributed independent variables with the probability law

$$p_r(\Delta_i) = 4rP(0)\, \mathrm{e}^{-4rP(0)\Delta_i} \qquad i = 2, 3. \tag{5.6}$$

Moreover, in the case (5.5), one necessarily has

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (+, -, -) \qquad \text{or} \qquad (-, +, +) \tag{5.7}$$

which carries a weight $\frac{1}{4}$. The expression (5.3) can be transformed into

$$\rho_r(\beta) \simeq \frac{1}{4\beta} \int_0^{+\infty} \mathrm{d}\Delta_2\, p_r(\Delta_2) \int_0^{+\infty} \mathrm{d}\Delta_3\, p_r(\Delta_3)(\beta\Delta_2\, \mathrm{e}^{-2\beta\Delta_2} + \beta(\Delta_2+\Delta_3)\, \mathrm{e}^{-2\beta(\Delta_2+\Delta_3)})$$
$$\times\, \delta(1 - \mathrm{e}^{-2\beta\Delta_2} - \mathrm{e}^{-2\beta(\Delta_2+\Delta_3)}). \tag{5.8}$$

In order to show that $\rho_r(\beta)$ is of the form (5.1), we put $u_i \equiv \beta\Delta_i$ $(i = 2, 3)$. Then $\beta\rho_r(\beta)$ becomes a function only of the variable α, through the probability law

$$p_r\left(\frac{u_i}{\beta}\right) \mathrm{d}\left(\frac{u_i}{\beta}\right) = 4\alpha\, \mathrm{e}^{-4\alpha u_i}\, \mathrm{d}u_i \qquad i = 2, 3. \tag{5.9}$$

We may obtain the leading order in the small-α expansion of $\mathscr{F}_-(\alpha)$ by expanding the exponential in (5.9) around $\alpha = 0$,

$$\mathscr{F}_-(\alpha) = 4\alpha^2 \int_0^{+\infty} \mathrm{d}u_2 \int_0^{+\infty} \mathrm{d}u_3 (u_2\, \mathrm{e}^{-2u_2} + (u_2 + u_3)\, \mathrm{e}^{-2(u_2+u_3)})$$
$$\times\, \delta(1 - \mathrm{e}^{-2u_2} - \mathrm{e}^{-2(u_2+u_3)}) + \mathrm{O}(\alpha^3). \tag{5.10}$$

The next order in the small-α expansion of (5.9) constitutes only one out of several contributions to the order α^3 terms appearing in (5.10), as one shall see later.

Integration on the variable u_3 of the leading term in (5.10) gives

$$\mathscr{F}_-(\alpha) \simeq 2\alpha^2 \int_0^{+\infty} \mathrm{d}u_2 \left(\frac{u_2\, \mathrm{e}^{-2u_2}}{1 - \mathrm{e}^{-2u_2}} - \tfrac{1}{2} \log(1 - \mathrm{e}^{-2u_2})\right)$$
$$= \frac{\alpha^2}{2} \left(\frac{\partial}{\partial y} - 1\right) \int_0^{+\infty} \mathrm{d}x\, \log(1 - \mathrm{e}^{-yx})\big|_{y=1}. \tag{5.11}$$

After performing this last integral we obtain the final result

$$\mathscr{F}_-(\alpha) \simeq \frac{\pi^2}{6}\, \alpha^2 \qquad \text{for } \alpha \ll 1. \tag{5.12}$$

A few comments are in order.

First of all, one easily sees that if one takes, instead of (5.5), the interval length Δ_4 of order β^{-1}, one obtains a contribution to the final result that is an order α smaller than (5.12).

Secondly, we have started out by taking Δ_1 of order β^0. One may verify that taking Δ_1 of order β^{-1} again contributes an extra factor α with respect to the leading term. By collecting in a similar way the relevant contributions to each order in α, it is possible, albeit somewhat complicated, to determine the small-α expansion of the scaling function $\mathscr{F}_-(\alpha)$. If we assume that this expansion converges for all $\alpha < \frac{1}{2}$, we have

$$\rho_r(\beta) = \frac{1}{\beta}\, \mathscr{F}_\pm(\alpha) \equiv \frac{1}{\beta}\, \mathscr{F}(\alpha) \qquad \text{for } \alpha \gtrless \tfrac{1}{2} \tag{5.13}$$

i.e. a scaling behaviour for arbitrary ratio r/β.

The calculation of this section has a clear physical interpretation. In the low-temperature limit, a frustrated loop will have the frustration localized on its weakest bond. When the temperature goes up, the frustration may prefer to be localized on the two next weakest bonds, which raises the energy but also the entropy. If these two bonds are not on the same semicircle connecting sites 0 and r as is the weakest bound (the condition (5.5)), then this shift of the frustration will involve the reversal of a string of spins including either s_0 or s_r, and result in a change of the sign of $\langle s_0 s_r \rangle_\beta$.

6. The total number of zeros of $\langle s_0 s_r \rangle_\beta$

The total number $\mathcal{N}_r$ of zeros of the correlation function $\langle s_0 s_r \rangle_\beta$ on the β-axis is

$$\mathcal{N}_r = \int_0^{+\infty} \mathrm{d}\beta\, \rho_r(\beta). \tag{6.1}$$

The results of the previous sections suggest to write ρ_r as

$$\rho_r(\beta) \simeq \begin{cases} \rho_\infty(\beta) & 0 \leqslant \beta \leqslant ar & (6.2a) \\ (1/\beta)\mathcal{F}(rP(0)/\beta) & ar < \beta < \infty & (6.2b) \end{cases}$$

where a is a fixed small enough number.

The integral (6.1) can be split up accordingly. The one on the interval $ar < \beta < \infty$ does not depend on r, as one sees upon replacing the variable of integration β by $x \equiv \beta / r$. The expression (6.1) therefore takes the form, for large r,

$$\mathcal{N}_r \simeq \int_0^{ar} \mathrm{d}\beta\, \rho_\infty(\beta). \tag{6.3}$$

We make use now of the large-β expansion of $\rho_\infty(\beta)$ given by (3.9), and integrate on β in (6.3). The result is that in the large-r limit

$$\mathcal{N}_r = \frac{c}{2\pi} \log r + \mathrm{O}(1) \tag{6.4}$$

with c given by (3.9a).

7. An application: a frustrated double chain

Exact results on correlation functions in random systems are relatively scarce compared with what is known about the thermodynamics of such systems. Therefore, as an application of the work of the preceding sections, we consider two infinite Ising chains whose spin variables, denoted $\{s_i\}$ and $\{s_i'\}$, respectively, are coupled by bonds of strengths $\{K_i\}$ and $\{K_i'\}$, respectively. Moreover, there will be couplings $\{J_i\}$ between corresponding spins of different chains, so that the Hamiltonian is

$$\mathcal{H} = \sum_{i=-\infty}^{\infty} (K_i s_{i-1} s_i + K_i' s_{i-1}' s_i' + J_i s_i s_i'). \tag{7.1}$$

As before, the K_i and K_i' will be identically distributed independent random variables, with a symmetric distribution P that we need not specify for the moment. For the J_i we postulate

$$J_i = \begin{cases} +\infty & \text{with probability } p/2 \\ -\infty & \text{with probability } p/2 \\ 0 & \text{with probability } 1-p. \end{cases} \tag{7.2}$$

Hence, on sites i that have $J_i = \infty$ or $J_i = -\infty$, we may set $s_i = s'_i$ or $s_i = -s'_i$, respectively, and the double chain becomes a chain of loops of different lengths (see figure 1).

Models of this kind have served before to study correlation functions in random systems; after the transformation $s_i s'_i = \sigma_{2i}$ and $s'_i s_{i+1} = \sigma_{2i+1}$, ours becomes equal to a random field chain considered by Grinstein and Mukamel [15], (see also [16]), but which has, moreover, random couplings.

The probability p_l of a loop of length $2l$ is

$$p_l = p(1-p)^{l-1} \qquad l = 1, 2, 3, \ldots. \tag{7.3}$$

We are interested in the correlation function of two spins far apart on the chain, and shall choose both of these for convenience on nodal points (sites with infinite interchain coupling). Let their distance be N, and let them be separated by L loops. The correlation $\langle s_0 s_N \rangle_\beta$ then is a product of L independent factors each coming from one loop. Hence, the density $\rho_p(N;\beta)$ of zeros of $\langle s_0 s_N \rangle_\beta$ is obtained after a disorder average that may be performed in two steps: first on the couplings K_i and K'_i; and then on the couplings J_i.

We consider the limit of a large number of loops, i.e. $L \gg 1$. Since from (7.3), we have that the average interval length is $1/p$, it follows that $L = Np$. We may write

$$\rho_p(N;\beta) = Np \sum_{l=1}^{\infty} p_l \rho_l(\beta) \qquad Np \gg 1. \tag{7.4}$$

In this expression $\rho_l(\beta)$ is the density of zeros of a loop of length l studied before.

In the small-p limit this expression can be made more explicit. The sum in (7.4) can then be replaced by an integral and, expanding in p, we obtain

$$\rho_p(N;\beta) \simeq Np^2 \int_0^{+\infty} \mathrm{d}l\, \mathrm{e}^{-pl} \rho_l(\beta). \tag{7.5}$$

The density $\rho_l(\beta)$ takes the two forms (6.2) which depend on the position of l relative to β. We consider the two β-intervals in (6.2) with r replaced by $1/p$, which is the characteristic value of l.

(i) When $\beta \ll 1/p$, the density $\rho_l(\beta)$ may be replaced by $\rho_\infty(\beta)$, and the integration in (7.5) gives

$$\rho_p(N;\beta) \simeq Np\rho_\infty(\beta). \tag{7.6a}$$

(i) When $\beta \sim 1/p$, we make use of the scaling function $\mathscr{F}(lP(0)/\beta)$. The transformation $x \equiv lP(0)/\beta$ in (7.5) yields

$$\rho(N;\beta) \simeq Np^2 P^{-1}(0)\mathscr{G}(\beta p/P(0)) \tag{7.6b}$$

where

$$\mathscr{G}(y) = \int_0^{+\infty} \mathrm{d}x\, \mathscr{F}(x)\, \mathrm{e}^{-xy}. \tag{7.7}$$

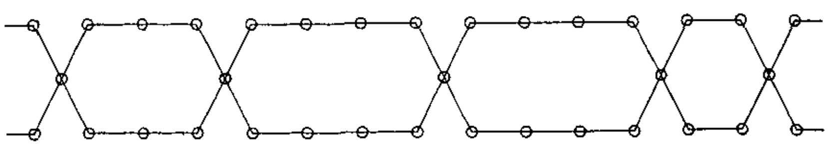

Figure 1. A chain of loops with random lengths. In the text, the correlation function of two spins located on nodal sites is studied.

We can ask what is the total number $\mathcal{N}_{p,N}$ of zeros of $\langle s_0 s_N \rangle_\beta$ in the double chain. The similarity of the equations (6.2) and (7.6) allows us to use the result (6.4) of the β-integration (6.1), replacing r by $1/p$. We obtain

$$\mathcal{N}_{p,N} = \frac{cNp}{2\pi} \log(1/p) \qquad p \ll 1,\ Np \gg 1 \tag{7.8}$$

which shows that this number is non-analytic in the point of zero frustration, $p = 0$.

8. Discussion

As a function of the inverse temperature β, the correlation $\langle s_0 s_r \rangle_\beta$ between two spins diametrically opposite in a random Ising loop of $2r$ spins, undergoes random sign changes with a density $\rho_r(\beta)$.

Several of the results concerning $\rho_r(\beta)$, and announced in the introduction, could easily be extended to the correlation $\langle s_0 s_n \rangle_\beta$ for spins at an arbitrary distance n. Furthermore, the method followed in the preceding sections could be used to calculate, for example, the distribution $\rho(\beta, \beta')$ of pairs of zeros.

The density $\rho_r(\beta)$ characterizes the random dependence of the equilibrium state of a frustrated system on the external parameters. A different characterization is provided by the correlation overlap function [2, 4–6, 12, 13]

$$G_r(\Delta\beta; \beta) = \overline{\langle s_0 s_r \rangle_\beta \langle s_0 s_r \rangle_{\beta+\Delta\beta}} \tag{8.1}$$

and it is of interest to discuss the connection between the two.

One expects, in zero field, that $G_r(\Delta\beta; \beta)$ falls off to zero with increasing $\Delta\beta$, and that its decay is faster as r is larger. Let $\Delta\beta_r(\beta)$, which we shall suppose to be much less than β, be the characteristic scale of this decay. (It is assumed here that such a scale exists, although in at least one case [2] the decay rather seems to be power-law-like.) We are therefore led to expect that for small $\Delta\beta_r$

$$\Delta\beta_r(\beta) \simeq 1/\rho_r(\beta). \tag{8.2}$$

The inverse $r_{\Delta\beta}$ of $\Delta\beta_r$ then is the *overlap length* between the two equilibrium states, at β and at $\beta + \Delta\beta$.

Within the droplet model, Fisher and Huse [3], and Bray and Moore [5], argue that for small $\Delta\beta$

$$r_{\Delta\beta} \sim |\Delta\beta|^{-2/(d_s - 2y)} \tag{8.3}$$

where $-y$ is the thermal exponent of the zero-temperature fixed point, d_s is the fractal dimension of the droplet surface area, and one has $d_s - 2y > 0$. From (8.2) and (8.3) it follows that

$$\rho_r(\beta) \sim r^{(d_s - 2y)/2} \qquad \text{for } r \gg 1. \tag{8.4}$$

In dimension $d = 1$, one has $d_s = 0$ and $y = -1$ so that $\rho_r(\beta) \sim r$. This is in agreement with what we found in the application of section 7 (where N plays the role of r here). Furthermore equation (8.4) shows that, in higher dimensions, a method of finding $\rho_r(\beta)$ as a function of r would constitute an independent way of determining the exponent $d_s - 2y$.

Acknowledgments

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Appendix. Derivation of (3.9*b*)

From (3.4) and (3.8) we obtain

$$\rho_\infty(\beta)=\frac{1}{2\pi\beta}\left(\frac{c_0c_2-c_1^2}{c_0^2}\right)^{1/2} \tag{A1}$$

where, using (2.14) for h and (2.8) for g, we have

$$c_n=\int_0^\infty \mathrm{d}x\left(\frac{2x}{\sinh 2x}\right)^n(-\log\tanh x)^{2-n} \qquad n=0,1,2. \tag{A2}$$

After transforming to the new variable of integration $y=-\log\tanh x$ we find

$$c_0=\tfrac{1}{2}\int_0^{+\infty}\mathrm{d}y\frac{y^2}{\sinh y}=\frac{7}{4}\zeta(3) \tag{A3a}$$

where we used Gradshteyn and Ryzhik [17, p 348]. Upon integrating by parts one sees that

$$c_1=-\tfrac{1}{2}\int_0^{+\infty}\frac{\mathrm{d}}{\mathrm{d}x}(\log\tanh x)^2x\,\mathrm{d}x=\tfrac{1}{2}c_0. \tag{A3b}$$

The coefficient c_2 may be rewritten as

$$c_2=\tfrac{1}{2}\int_0^{+\infty}\mathrm{d}y\left(\frac{y}{\sinh y}\right)^2=\tfrac{1}{2}\zeta(2)=\frac{\pi^2}{12} \tag{A3c}$$

(see [17, p 352]). By combining (A.1) with (A.3) one obtains (3.9).

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